Level-rank duality in nonunitary coset theories

1990 J. Phys. A: Math. Gen. 23 L789

(http://iopscience.iop.org/0305-4470/23/16/002)

J. Phys. A: Math. Gen. 23 (1990) L789–L793. Printed in the UK

LETTER TO THE EDITOR

Level-rank duality in non-unitary coset theories

Daniel Altschuler, Michel Bauer and Hubert Saleur

Service de Physique Théorique†, CEN-Saclay, F-91911 Gif-sur-Yvette Cédex, France

Received 11 June 1990

Abstract. It is shown that the conformal field theories $SU(n)_k \times SU(n)_1/SU(n)_{k+1}$ and $SU(m)_l \times SU(m)_1/SU(m)_{l+1}$ are equivalent, where $k+n=n/m$, $l+m=m/n$, for a pair (m, n) of relatively prime positive integers.

We will prove a new equivalence theorem concerning the non-unitary coset models $SU(n)_k \times SU(n)_1/SU(n)_{k+1}$ with a rational level $k=t/m$. These theories can be interpreted as representations of the W_n algebra [1] but this will not be important for us in what follows. Their central charge is:

$$c_{r,s} = (n-1)\left(1-\frac{n(n+1)(s-r)^2}{rs}\right) \tag{1}$$

where r, s are positive coprime integers related to k and n as follows: $r=m(k+n)$, $s=m(k+n+1)$. Call these models $M_{r,s}$ for short. Their branching functions are parametrised by a pair (a, b) of dominant *integral* weights of level s, r respectively, i.e.

$$a = a_0\Lambda_0 + a_1\Lambda_1 + \ldots + a_{n-1}\Lambda_{n-1} \tag{2}$$

with Λ_i the fundamental weights of the affine algebra of $SU(n)$, such that each a_i is a strictly positive integer, and $\Sigma_{i=0}^{n-1} a_i = s$, and similarly for b. Explicitly these functions are given by $\chi_{a,b}(q) = \Delta_{a,b}(q)\eta(q)^{1-n}$ where:

$$\Delta_{a,b}(q) = \sum_{\alpha\in Q}\sum_{w\in W_0} \varepsilon(w) q^{(1/2)rs|\alpha+w(a)/s-b/r|^2} \tag{3}$$

Q is the root lattice, and W_0 is the finite Weyl group of $SU(n)$. This formula has been established several times already in the literature. It is easily derived from the character formulae for $SU(n)_k$ [2].

It is believed that these coset theories are the critical limit of RSOS models [3]. This was further elaborated in [4]. There it is shown that the two operations: taking q into $-q^{-1}$, where now $q=\exp(i\pi r/s)$ is the deformation parameter of the quantum group $sl_q(n)$, or exchanging $s-n$ and n (in both cases the spectral parameter u is simultaneously negated) transform a given RSOS model to an equivalent one.

The latter operation is now called *level-rank duality*. Two of us recently wrote about the meaning of this symmetry, mostly in the language of quantum groups and their commutants [5]. The history of the subject goes back at least to [6]. So far this duality has manifested itself in several ways. The first one has its explanation in terms of conformal embeddings [6–9]. The second is the duality of RSOS models noticed in

† Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique.

[10]. This version extends to fusioned RSOS (face) models as argued in [5, 11], where in the latter paper it is also proved in the course of the arguments that the values of the fusion coefficients in the Verlinde algebra of the WZW models $SU(n)_m$ and $SU(m)_n$ coincide. This is the still embryonic third manifestation of level-rank duality† in WZW models. It is exciting to think that in due course one should expect a Chern–Simons (3D) geometric explanation of it. We have already obtained [5] relations between the knot invariants from the quantum group formalism. In this paper we discuss the most recent [4] fourth manifestation, occurring between the above-mentioned non-unitary coset models (r, s) when $r = n$ and $s = m + n$. As always we obtain symmetry under interchange of m and n, but this time the meaning is equivalence of the models $M_{n,m+n}$ and $M_{m,m+n}$. The authors of [4] found it as a by-product of their analysis of the regimes of the non-unitary RSOS models. Thus they explained the fact that the dimensions d of the conserved charges of the rational Toda field theories, obtained from the perturbation of these coset models, are characterised by $\gcd(d, m, n) = 1$. They noticed that the central charge (1) when $r = n$ becomes:

$$c_{n,m+n} = -\frac{(n-1)(m-1)(m+n+mn)}{m+n} \tag{4}$$

which is symmetric in m and n. If a conjecture [3] on the local state probabilities (one-point function) of the non-unitary RSOS models is true, it implies the equality of the branching functions (3) upon permuting m with n. This last equality was then checked explicitly for the first non-trivial example which is the Lee–Yang singularity $(m, n) = (2, 3)$. In this note we shall show that this equality among branching functions holds for general (m, n) directly by manipulating the expression (3). We wish to thank the authors of [4] for communicating to us a preliminary version of their work.

We begin as in [3] by the observation that (3) can be written more compactly as:

$$\Delta_{a,b}(q) = q^{b^2/2(k+n) - a^2/2(k+n+1)} \sum_{w \in W} \varepsilon(w) q^{|w(a)-b|^2/2} \tag{5}$$

where now summation is over the full affine Weyl group W. Now if $r = n$ we have no choice but to set $b = \rho = \Sigma_{i=1}^{n-1} \Lambda_i$. Taking this into account and expanding the power of q inside the sum we get:

$$q^h \Delta_{a,\rho}(q) = q^{\langle \rho, a \rangle} \sum_{w \in W} \varepsilon(w) q^{-\langle \rho, w(a) \rangle} = \prod_{\alpha \in \Sigma_+} (1 - q^{\langle \alpha, a \rangle})^{\mathrm{mult}(\alpha)} \tag{6}$$

using the denominator identity, where h is some rational number. Here Σ_+ is the set of positive roots of the affine Lie algebra of $SU(n)$. We denote the product side of (6) by $\Delta_a^{(n)}(q)$, as we want to emphasise the n dependence. What we show is that

$$\Delta_a^{(n)}(q) \varphi^m(q) = \Delta_{t(a)}^{(m)}(q) \varphi^n(q) \tag{7}$$

where $\varphi(q) = \Pi_{j=1}^{\infty} (1 - q^j)$ is the inverse of Euler's generating function of partitions. Here $t(a)$ means the $SU(m)$ weight obtained by transposing along the NW-SE diagonal the Young tableau of $a - \rho^{(n)}$ and adding $\rho^{(m)}$ to the result. Note that the prefactors q^h will automatically coincide as a consequence of (7) because the $\chi_{a,b}$ are modular functions.

† While completing this paper we received [16] in which consequences of the duality for WZW models are analysed.

Clearly (7) is equivalent to the equality of the characters $\chi_a^{(n)}(q)$ and $\chi_{t(a)}^{(m)}(q)$. At this point we note that (6) is nothing but the numerator of the character of the representation of $\mathrm{SU}(n)_m$ with highest weight $a-\rho$ evaluated in the principal specialisation. Now the identity (7) was discovered a long time ago [12]. In order to make this letter self-contained, we decided however to present our own proof, which relies mainly on the tools developed in [9].

Recall that Σ_+ is the set of vectors of the form $e_i - e_j + l\delta$ where $\{e_i\}_{1\leqslant i\leqslant n}$ is an orthonormal basis of n-dimensional Euclidean space, δ is the null root, and we require that $l>0$ if $i\leqslant j$, $l\geqslant 0$ if $i>j$. In this basis the expansion of a is:

$$a=\sum_{i=1}^{n} r_i e_i - \frac{1}{n}\left(\sum_{i=1}^{n} r_i\right)\left(\sum_{i=1}^{n} e_i\right) \tag{8}$$

and moreover $m+n=r_1>r_2>\ldots>r_n\geqslant 1$. Thus the left-hand side of (7) yields:

$$\prod_{l=0 \text{ or } 1}^{\infty} \prod_{i,j=1}^{n} (1-q^{l(m+n)+r_i-r_j}) \prod_{l=1}^{\infty} (1-q^l)^m \tag{9}$$

and the right-hand side is similar, involving the coordinates of $t(a): m+n=s_1>s_2>\ldots>s_m\geqslant 1$ instead of the r_i.

Consider the cyclic group Z_n of n elements which acts on the weights of $\mathrm{SU}(n)$ by rotating the Dynkin coordinates. We represent the Z_n orbit of a on figure 1 as follows. A circle is divided into $m+n$ arcs of equal length. Corresponding to the Dynkin coordinates $(a_1, a_2, \ldots, a_{n-1}, a_n \equiv a_0)$ we cut slices of sizes $a_1, \ldots, a_n$ counter-clockwise around the circle. The Z_m orbit of $t(a)$ corresponds to the complementary slicing indicated by broken lines, which is read in the opposite sense (clockwise). Since

$$r_i=\sum_{j=i}^{n} a_j \tag{10}$$

we see that with l fixed, by varying i, j in $r_i - r_j$ we obtain in turn all the coordinates of all the elements in the orbit of a. For $p\in\{1,\ldots,m+n\}$ let $\mu_a(p)$ be the multiplicity of p among the coordinates of the elements of the orbit of a. Then (7) is equivalent to:

$$\mu_a(p)=\mu_{t(a)}(p)+n-m. \tag{11}$$

To prove this, suppose A is a subset of the additive group Z_{m+n}. We denote by $\bar{A}$ its complementary subset, by $A+z$, where $z\in Z_{m+n}$, its z-translate and by $-A$ the opposite.

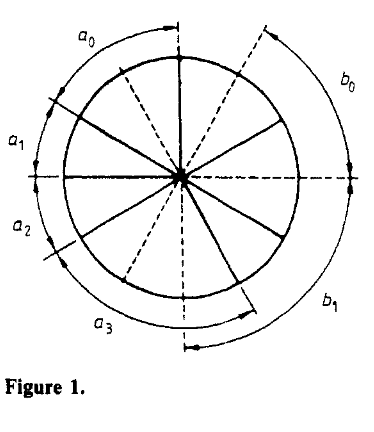

Figure 1.

We identify Z_{m+n} with $\{1, \ldots, m+n\}$ by setting $m+n \equiv 0$. Then if $A = \{r_1, \ldots, r_n\}$, by combining these three operations a subset corresponding to $t(a)$ is obtained. We introduce the characteristic function of A:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases} \tag{12}$$

We fix a primitive $(m+n)$th root of unity ω, and consider the finite Fourier transform of χ_A:

$$\psi_A(y) = \sum_{x \in A} \omega^{xy}. \tag{13}$$

We notice that

$$|\psi_{A+z}(y)|^2 = |\psi_{-A}(y)|^2 = |\psi_A(y)|^2 \tag{14}$$

and also

$$\psi_A(y) + \psi_{\bar{A}}(y) = \begin{cases} 0 & y \neq 0 \\ m+n & y = 0. \end{cases} \tag{15}$$

Furthermore, $\psi_A(0) = \text{card}(A) = n$ and $\psi_{\bar{A}}(0) = \text{card}(\bar{A}) = m$. The crucial property of ψ_A is that $\mu_a(x) = \Gamma_A(x)$ where

$$\Gamma_A(x) = \frac{1}{m+n} \sum_{y=0}^{m+n-1} |\psi_A(y)|^2 \omega^{-xy}. \tag{16}$$

We rewrite this as:

$$\Gamma_A(x) = \frac{n^2}{m+n} + \frac{1}{m+n} \sum_{y \neq 0} |\psi_A(y)|^2 \omega^{-xy} \tag{17}$$

and since, if $y \neq 0$, $|\psi_A(y)|^2 = |\psi_{\bar{A}}(y)|^2$, we find:

$$\Gamma_A(x) = \frac{n^2 - m^2}{m+n} + \Gamma_{\bar{A}}(x) \tag{18}$$

thereby completing the proof of (11).

To conclude, a few physical remarks are in order. The duality of RSOS models is naturally expected to give rise to duality between some conformal field theories (CFT). Indeed, the lattice models considered having trigonometric Boltzmann weights are critical, and the equality of their discrete cylinder partition functions (with proper mapping of boundary conditions [5]) translates after taking the continuum limit into an equality of the characters of their associated CFT. The problem however is that dual RSOS models have opposite spectral parameters, and in one case at least u lies in a regime which is difficult to analyse.

To illustrate this consider the RSOS model whose local states are dominant integral weights of SU(n) at level two. Let $P_{n,2}$ be the set of these states. Set $q = \exp(i\pi/n+2)$. For $u > 0$ (transition from regime III to IV [13]) it is known that the associated CFT is $\mathrm{SU}(n)_1 \times \mathrm{SU}(n)_1/\mathrm{SU}(n)_2$. After the duality transformation we get the $P_{2,n}$ models with the same q but $u < 0$ (transition from regime I to II). The derivation of the continuum limit CFT is now more involved. A subtle cancellation mechanism [13, 14] produces another coset, $\mathrm{SU}(2)_n/\mathrm{U}(1)$. Hence the equality of SU(2) string functions and $\mathrm{SU}(n)_1 \times \mathrm{SU}(n)_1/\mathrm{SU}(n)_2$ branching functions [8] is in fact also a form of level-rank duality.

As another example take the $P_{2,n}$ RSOS models, but with $q=\exp(\mathrm{i}\pi m/n+2)$, $m\neq 1$. For $u>0$ the associated CFT is $SU(2)_k\times SU(2)_1/SU(2)_{k+1}$, where $k=(n+2)m^{-1}-3$. By duality it transforms to $P_{n,2}$ with the same q and $-u$, or equivalently $-q^{-1}$ and u. The usual analysis [15] suggests $SU(n)_l\times SU(n)_1/SU(n)_{l+1}$ as the continuum limit, with $l=(n+2)m^{-1}-n-1$. This would imply the equality of the corresponding characters, which is clearly not true in general. Only for $m=2$, a particular case of what is treated above, do characters correspond. For $m\neq 1$, 2 the continuum limit of $P_{n,2}$ models thus seems to require further study. We hope it lead to other nice identities between characters of different (dual) CFT.

We wish to thank J-B Zuber for a careful reading of the manuscript.

References

[1] Fateev V A and Zamolodchikov A B 1987 *Nucl. Phys.* B **280** 644
Fateev V A and Lykyanov S L 1988 *Int. J. Mod. Phys.* A **3** 507
[2] Kac V and Wakimoto M 1988 *Proc. Natl Acad. Sci. USA* **85** 4956
[3] Nakanishi T 1990 *Nucl. Phys.* B **334** 745
[4] Kuniba A, Nakanishi T and Suzuki J 1990 Ferro and antiferro-magnetizations in RSOS models *Preprint*
[5] Saleur H and Altschuler D 1990 Level-rank duality in quantum groups *Preprint* Saclay SPhT/90-041
[6] Jimbo M and Miwa T 1984 *Adv. Stud. Pure Math.* **4** 97
Jimbo M and Miwa T 1985 *Adv. Stud. Pure Math.* **6** 17
[7] Sasaki R and Yamanaka I 1986 *Prog. Theor. Phys.* **75** 706
[8] Altschuler D 1989 *Nucl. Phys.* B **313** 293
[9] Altschuler D, Bauer M and Itzykson C 1990 The branching rules of conformal embeddings *Commun. Math. Phys.* to appear
[10] Jimbo M, Miwa T and Okado M 1987 *Lett. Math. Phys.* **14** 123
[11] Kuniba A and Nakanishi T 1990 Level-rank duality in fusion RSOS models *Proc. Int. Colloq. on Modern Quantum Field Theory (Bombay)* in press
[12] Frenkel I B 1982 *Lecture Notes in Mathematics* vol 933 (Berlin: Springer) p 71
[13] Andrews G E, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193
[14] Saleur H 1990 Zeroes of chromatic polynomials, a new approach to the Beraha conjecture using quantum groups *Commun. Math. Phys.* to appear
[15] Kostov I 1988 *Nucl. Phys.* B **300** 559
[16] Naculich S G and Schnitzer H J 1990 *Preprints* Brandeis BRX-TH-288 and 289
Naculich S G, Riggs H A and Schnitzer H J 1990 *Preprint* Brandeis BRX-TH-293